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Nonlinear gravity waves on steady non-uniform currents

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The interaction of nonlinear gravity waves and steady non-uniform currents is studied using the averaged Lagrangian method due to Whitham (1965*a*,*b*). The results are compared with the essentially linear theory of Longuet-Higgins & Stewart (1961, 1964) for three specific problems: waves on a stream $(U(x), 0)$ with variations in the stream balanced by upwelling from below or inflow from the sides, and waves on a shear flow $(0, V(x))$. It appears that rates of growth of large waves are less than those predicted by linear theory and that the energy density can sometimes decrease when the wave height and steepness are still increasing. The final section discusses the form of the energy equation in terms of the Lagrangian.

1. Introduction

In a series of recent papers Whitham (1965*a*,*b*, 1967) has provided a new technique for investigating the properties of nonlinear dispersive wave systems and has followed this up by applying the method to gravity surface waves in uniform conditions. However, the method applies equally well to slowly varying non-uniform conditions, but so far there have been few attempts to produce results from it in this area. The present author has considered capillary waves on a non-uniform stream (Crapper 1970) and the object of this paper is to apply similar ideas to gravity waves in deep water.

The relevant earlier work on this subject is contained in papers by Longuet-Higgins & Stewart (1961, 1964). In these a perturbation analysis to second order in the gravity wave is used to find the correct amplitude variation, and this is expressed in a modified energy equation which includes a 'radiation-stress' term expressing the interaction between the waves and the current. We shall see here that by using Whitham's theory their result comes out directly from a knowledge of linear gravity-wave theory and we shall go on to consider the fully nonlinear case. We shall also consider the form of the energy equation in terms of Whitham's averaged Lagrangian density.

The results given here are computed from a nonlinear ordinary differential equation and are presented in a series of figures. The differences from the previous linear solution are discussed in § 4. The most surprising feature is the behaviour of the energy density function, which in certain cases actually starts to decrease when the waves are still increasing in height and steepness. This would appear to be a result of the fact that, as Whitham shows, changes in energy propagate at a different speed from changes in wavelength in a nonlinear system, although in a linear system both these speeds reduce to the group velocity.

2. The linear solution

Whitham's averaged Lagrangian density $\mathscr{L}$ can be expressed as the difference between the mean kinetic and potential energies, $\mathscr{T}-\mathscr{V}$, and for ordinary linear gravity waves in deep water with no basic flow

$$\mathscr{L} = \tfrac{1}{4}\rho g a^2(\omega^2/gk-1). \tag{1}$$

Here $2a = H$ is the height between trough and crest, ω the frequency and k the wavenumber. We wish to apply the method to waves on a running stream. Lighthill (1967) has shown that for steady waves on a running stream with horizontal components $(U(x), 0)$ all that is necessary is to replace ω in (1) by $-Uk$. The logical move, therefore, for unsteady waves travelling in the x direction on the same stream is to replace ω by $\omega - Uk$, giving

$$\mathscr{L} = \tfrac{1}{4}\rho g a^2[(\omega-Uk)^2/gk-1]. \tag{2}$$

Whitham's equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right)-\frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial k}\right) = 0 \tag{3}$$

for the unsteady problem, but in the problem we wish to consider $\partial/\partial t \equiv 0$. The dynamical problem is unsteady, with the waves and moving across the stream, but the averaged system is steady, ω and k being functions of x only. Thus

$$\partial\mathscr{L}/\partial k = \text{constant}, \tag{4}$$

or, using (2),

$$a^2\left\{\frac{(\omega-Uk)^2}{k^2}+\frac{2U}{k}(\omega-Uk)\right\} = \text{constant}. \tag{5}$$

We also have the equation

$$\omega = k(c+U) = \text{constant}, \tag{6}$$

where c is the phase velocity of the gravity waves of wavelength $\lambda = 2\pi/k$ when there is no stream present. Equation (6) says that the actual speed of the waves in space is $c+U$; the fact that $\omega = $ constant is essentially an expression of the conservation of waves, for which Whitham's equation is

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial x} = 0, \tag{7}$$

and here $\partial k/\partial t \equiv 0$. Thus (5) becomes

$$a^2 c(c+2U) = \text{constant}, \tag{8}$$

which is the result of Longuet-Higgins & Stewart for the case where the continuity of the main stream is maintained by fluid upwelling from below. Their result for the case of a laterally converging current with no upwelling is slightly more complicated and we shall consider only the nonlinear version in the next section.

3. Nonlinear solutions

The first problem in applying Whitham's method is the choice of a suitable Lagrangian $\mathscr{L}$, which, since exact uniform wave train solutions are not known, must be some approximation. Lighthill (1967) has produced an approximate Lagrangian based on the first two terms of the series solution, the Stokes wave (Stokes 1847) and another contribution from the wave of greatest height (Michell 1893). This seems an appropriate choice for the present work and is given by

$$\mathscr{L} = (\rho g/8k^2)\,P(z), \tag{9}$$

where

$$P(z) = (z-1)^2-(z-1)^3-(z-1)^4, \tag{10}$$

$$z = \omega^2/gk = 1+\pi^2 s^2 \tag{11}$$

and $s = H/\lambda$ is the wave steepness. We now consider a stream with horizontal components (U, V) and waves with wavenumber $\mathbf{k} = (l, m)$, and replace ω in (11) by $\omega - Ul - Vm$. Whitham's equation is

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathscr{L}}{\partial l}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \mathscr{L}}{\partial m}\right) = 0, \tag{12}$$

with

$$\omega = kc+Ul+Vm = \text{constant} \tag{13}$$

corresponding to (6). Also

$$\partial l/\partial y = \partial m/\partial x, \tag{14}$$

another consequence of the conservation of waves also shown by Whitham.

At this stage we introduce a system of non-dimensional variables $k^* = k/k_0$, $c^* = c/c_0$, $U^* = U/c_0$, $V^* = V/c_0$, where k_0 and c_0 are the values of k and c where U and V vanish; we shall drop the asterisks from now on. The most convenient variable is the phase velocity c, therefore after all differentiations with respect to l and m have been carried out we substitute kc for $\omega - Ul - Vm$ and $z_0 c^2 k$ for z, where $z_0 = c_0^2 k_0/g$. Then (12) becomes

$$\frac{\partial}{\partial x}\left\{\frac{2P\cos\theta}{k^3}+\frac{z_0 c}{k^2}(c\cos\theta+2U)\frac{dP}{dz}\right\} + \frac{\partial}{\partial y}\left\{\frac{2P\sin\theta}{k^3}+\frac{z_0 c}{k^2}(c\sin\theta+2V)\frac{dP}{dz}\right\} = 0, \tag{15}$$

where $(l, m) = (k\cos\theta, k\sin\theta)$ and from the non-dimensional form of (13)

$$k = 1/(c+U\cos\theta+V\sin\theta). \tag{16}$$

We shall consider three specific problems: (i) when the stream is in the x direction with continuity maintained by upwelling from below; (ii) when the stream is mainly in the x direction but with continuity maintained entirely by lateral inflow from the sides; (iii) with waves crossing a shear flow $(0, V(x))$.

In problem (i) we have $V \equiv 0$ and $\partial/\partial y \equiv 0$. If we also take $\theta \equiv 0$ we have the problem just considered in the linear case. Then (15) can be integrated to give

$$\frac{1}{k^2}\left\{\frac{2P}{k}+z_0 c(c+2U)\frac{dP}{dz}\right\} = \text{constant}. \tag{17}$$

Substituting for k from (16) gives a seventh-order polynomial for c which can be solved numerically and the root for which $c \to 1$ as $U \to 0$ chosen in agreement with our non-dimensionalization. However in view of problem (ii) it is more convenient to use (15) directly, which amounts to differentiating (17) with respect to x to give an equation of the form

$$A\frac{dc}{dx}+B\frac{dU}{dx} = 0 \quad \text{or} \quad \frac{dc}{dU} = -\frac{B}{A}, \tag{18}$$

where A and B are functions of c and U. At $U = 0$, $c = 1$ and if z_0 is specified by a suitable choice of initial steepness s_0 the equation can be solved numerically for $c(U)$ using standard library procedures. Then k follows from (16) and, when z has been calculated, s follows from (11).

If we allow the waves to make an angle with the x axis we still have $V \equiv 0$ and $\partial/\partial y \equiv 0$. Then (14) gives

$$m = k \sin\theta = \text{constant} = \sin\theta_0, \tag{19}$$

where θ_0 is the angle between the wavenumber vector and the x axis where $U = 0$. Using (19) and (16) we have

$$\sin\theta = m(c+U\cos\theta), \tag{20}$$

or

$$\sin\theta = \frac{mc \pm mU[1-m^2(c^2-U^2)]^{\frac{1}{2}}}{1+m^2U^2}. \tag{21}$$

Comparison with (20) shows that the sign should be chosen to be that of $\cos\theta$, which presents no difficulties, and thus all terms in (15) are again known as functions of c and U, leading to another equation of the same type as (18), of which, of course, the first is a special case.

For problem (ii) we have an additional term from the $\partial/\partial y$ bracket in (15) because now $dV/dy \neq 0$. We assume, however, that $\theta \equiv 0$ (waves in the x direction only) and that only V depends on y, with

$$dV/dy = -dU/dx. \tag{22}$$

This follows Longuet-Higgins & Stewart in the linear case, essentially solving only along the line of symmetry $y = 0$ and so putting $V = 0$, but not $dV/dy = 0$. With the extra term the polynomial solution is impossible, but the equation of type (18) which can be derived differs in only one term from the previous case and can be conveniently worked into the same program.

Finally, for problem (iii) we again assume $\partial/\partial y \equiv 0$ and now $U \equiv 0$ also. We again have (19), which with (16) gives

$$\sin\theta = mc/(1-mV), \tag{23}$$

and we obtain an equation of the form (18) for dc/dV.

4. Results

The results are presented graphically. Figures 1–5 show the results of calculations for problems (i) and (ii) by solid lines and broken lines respectively. The first three figures refer to the case where $\theta \equiv 0$ always. By considering H/H_0,

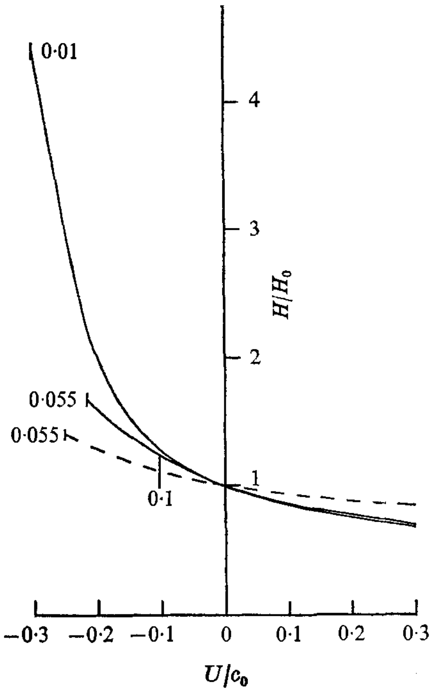

FIGURE 1. Wave height H as a function of stream velocity U. ——, problem (i), where variations of U are balanced by upwelling from below; – – –, problem (ii), where variations are made up by inflow from the side. Values of the initial wave steepness s_0 are shown by the curves.

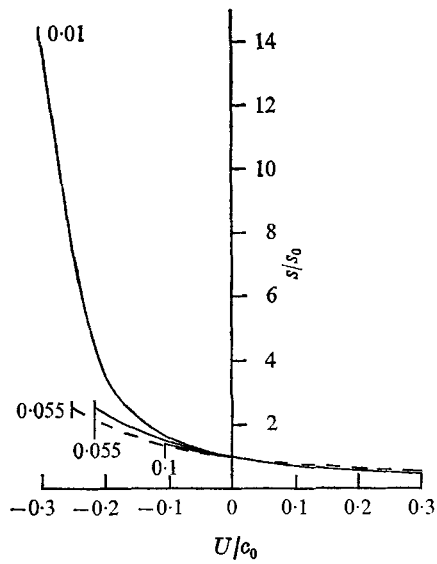

FIGURE 2. Wave steepness s as a function of stream velocity. Notation as in figure 1.

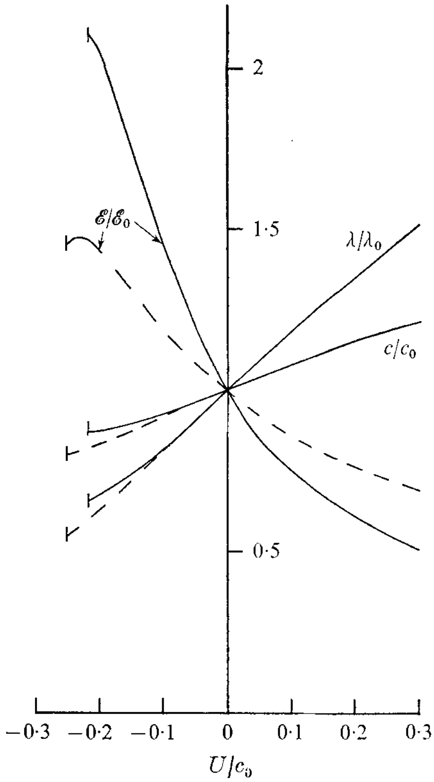

FIGURE 3. Wavelength λ, wave velocity c and energy density $\mathscr{E}$ as functions of stream velocity for initial steepness $s_0 = 0{\cdot}055$. ——, problem (i); – – –, problem (ii).

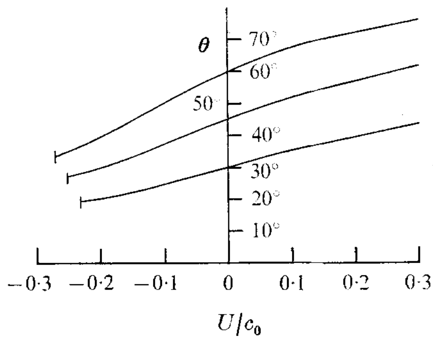

FIGURE 4. Angle θ as a function of stream velocity for problem (i) with initial values $\theta_0 = 30°$, 45° and 60° and initial steepness $s_0 = 0{\cdot}055$.

shown in figure 1, we see from the solid curves that the value of the initial steepness $s_0 = H_0/\lambda_0$ (the value where $U = 0$) has a noticeable effect on the growth of the wave height for $U < 0$, the smaller waves growing more rapidly than the medium and large waves, which are almost indistinguishable. If we consider the linear results of Longuet-Higgins & Stewart (1961, figure 1) we find that the curve is almost identical with our curve for $s_0 = 0{\cdot}01$ to the right of $U \simeq 0{\cdot}2$, where $H/H_0 = 2$, but rises more steeply to the left of this point. At $U = -0{\cdot}2$ the

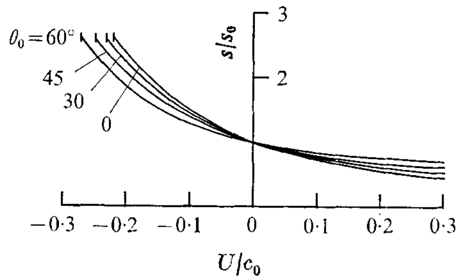

FIGURE 5. Wave steepness s as a function of stream velocity for problem (i) with $\theta_0 = 0°$, 30°, 45° and 60°, and $s_0 = 0{\cdot}055$.

wave steepness $s = 0{\cdot}037$ (from figure 2), a point we shall return to shortly. The problem (ii) curve shows a much less rapidly rising curve, as does the linear solution, but again the nonlinearity reduces the rate of growth. In the region $U > 0$ nonlinearity appears to have little effect in either problem, presumably because the waves are rapidly reduced to linear proportions. Figure 2 shows the corresponding curves for the wave steepness s; because of differences in the changes in λ the curves for $s_0 = 0{\cdot}055$ and $s_0 = 0{\cdot}1$ are more easily distinguished. The waves break when $s = 0{\cdot}142$ (Michell 1893), at which point all curves are terminated (in all the figures). Figures 4 and 5 show the effects in problem (i) of waves at an angle to the flow. As can be seen the effect on the steepness is surprisingly small.

Perhaps more interesting is figure 3, which shows λ/λ_0, c/c_0 and $\mathscr{E}/\mathscr{E}_0$, where $\mathscr{E}$ is the energy density, for the intermediate initial steepness $s_0 = 0{\cdot}055$. It will be noted that problems (i) and (ii) produce different values of λ/λ_0 and c/c_0 when U is sufficiently negative; in the linear solution the values are the same. The behaviour of the energy density is surprising, especially in problem (ii), where it is seen to begin to decrease before the waves break, on the left of the figure. The function used for the energy density is given by Whitham's result

$$\mathscr{E} = \omega \partial \mathscr{L}/\partial \omega - \mathscr{L} \tag{24}$$

(Whitham 1965*b*, equation (39), with the β and γ terms absent), with ω replaced by $\omega - Ul - Vm$. This gives

$$\mathscr{E} = \frac{\rho g}{8k^2}\left(2z\frac{dP}{dz} - P\right), \tag{25}$$

where the k is a dimensional quantity. Substituting for z in terms of s gives

$$\mathscr{E} = \tfrac{1}{2}\rho g(\pi^2 s^2/k^2)\,(1 - \tfrac{3}{4}\pi^2 s^2 - \tfrac{13}{4}\pi^4 s^4 - \tfrac{7}{4}\pi^6 s^6), \tag{26}$$

where the leading term is simply the linear result $\frac{1}{2}\rho g a^2$ with $a = 2H = \pi s/k$. The functions $8k^2\mathscr{E}/\rho g$ from (26) and the linear form (broken line) are shown in figure 6, along with the function $\mathscr{E}/\mathscr{M}c$,

$$\mathscr{M} = \frac{\rho g z}{8k^2 c}\frac{dP}{dz} \tag{27}$$

being the momentum density, which Whitham (1965*b*, equation (40)) shows to have components $(l\,\partial\mathscr{L}/\partial\omega, m\,\partial\mathscr{L}/\partial\omega)$. In linear solutions this function is identically

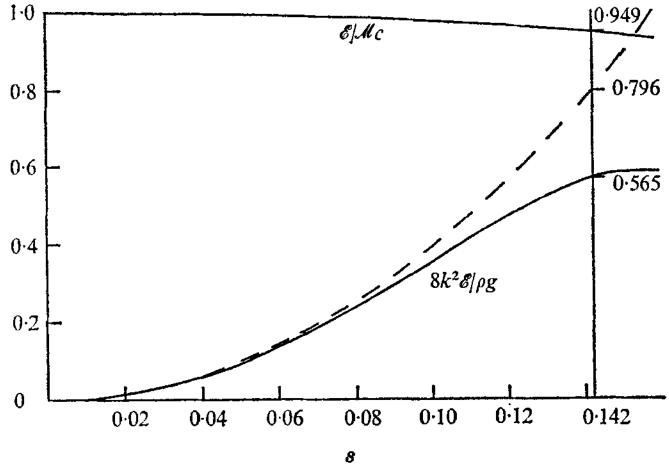

FIGURE 6. Energy functions $8k^2\mathscr{E}/\rho g$ for nonlinear (solid line) and linear (broken line) cases and the nonlinear function $\mathscr{E}/\mathscr{M}c$ as functions of wave steepness s.

equal to one. Two points are noticeable. First, the linear and nonlinear energy functions begin to move apart near $s = 0{\cdot}037$, the value noted earlier at which nonlinearity began to make itself felt. Second, the difference between the curves is such that, if k is increasing sufficiently rapidly as s increases, it is clearly possible for the linear $\mathscr{E}$ (and hence H) to increase whilst the nonlinear $\mathscr{E}$ decreases, as happens in figure 3. This can be regarded as an example of the 'splitting' of the group velocity, as noted by Whitham, so that changes in energy and changes in wavenumber propagate at different speeds. It could be argued that the reduction in $\mathscr{E}$ before breaking is a consequence of the approximate Lagrangian used, but as the Lagrangian was constructed with particular reference to the highest waves this does not seem very likely, and the curve for $\mathscr{E}$ must in any case be close to that shown in figure 6. The function $\mathscr{E}/\mathscr{M}c$ shows that although $\mathscr{M}$ decreases from its linear value the effect is not quite as large as it is on the energy. For the same c and k, $\mathscr{M}$ is 75 % of its linear value at breaking, whereas $\mathscr{E}$ is 71 %.

When we consider problem (iii), waves crossing a shear flow, the effect of the nonlinearity is in some ways less marked. Figures 7–10 show results for a single initial steepness, $s_0 = 0{\cdot}055$, and three initial angles, $\theta_0 = 30°$, $45°$ and $60°$. Calculations for other values of s_0 produce curves which are too close to these curves to be separated on the scales possible in print, although, of course, waves break at different points. All waves break at the points where the curves end on the left (for $V < 0$), but the curves end on the right (for $V > 0$) where θ reaches 90°. The curves of H/H_0 are very similar to the linear solution (Longuet-Higgins & Stewart 1961, figure 3) but terminate at finite values as $\theta \rightarrow 90°$. Indeed we can see that k, c and hence z, s, H and $\mathscr{E}$ all have finite values at this point from (23), which becomes $c = 1/m - V$, and (19), which is $k = m$, and the fact that V is finite. The linear theory has a caustic for these values of x and infinite energy. It will be noted that for this initial steepness the waves do not break before this point. The waves are reflected back if V gets sufficiently large

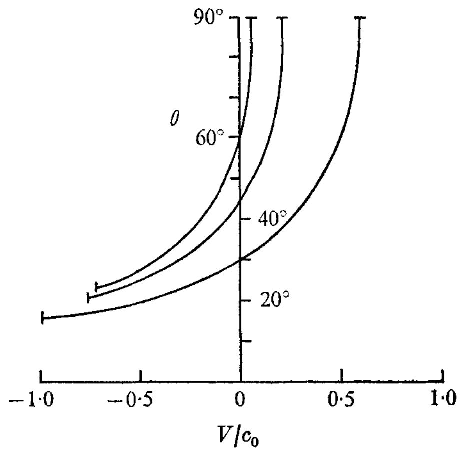

FIGURE 7. Angle θ as a function of stream velocity for problem (iii) (waves crossing a shear flow) with initial values $\theta_0 = 30°$, $45°$ and $60°$ and initial steepness $s_0 = 0{\cdot}055$.

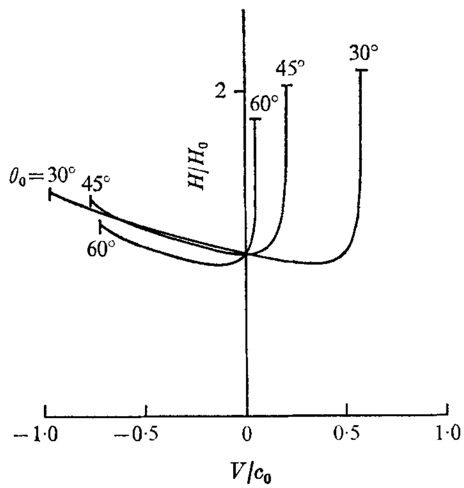

FIGURE 8. Wave height H as a function of stream velocity for the same situations as in figure 7.

to allow θ to reach 90°, and if this happens our results are not valid as the incident and reflected waves will interact with each other as well as with the stream. Also we must consider whether the assumption of slow variations which underlies the whole method still holds where θ approaches 90°. The rates of change of s are large as functions of V, but the wavelength is increasing and changes over one wavelength need not be large if V does not vary too rapidly, so the results may hold even in this region. Finally, we note that for $V < 0$ the energy density again begins to decrease before the waves break, whilst their height and steepness are still increasing.

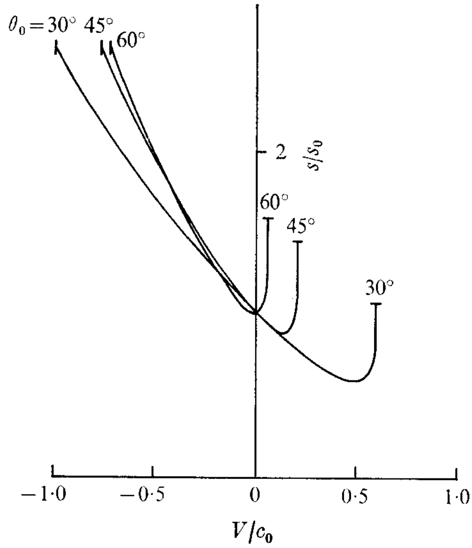

FIGURE 9. Wave steepness s as a function of stream velocity for the same situations as in figure 7.

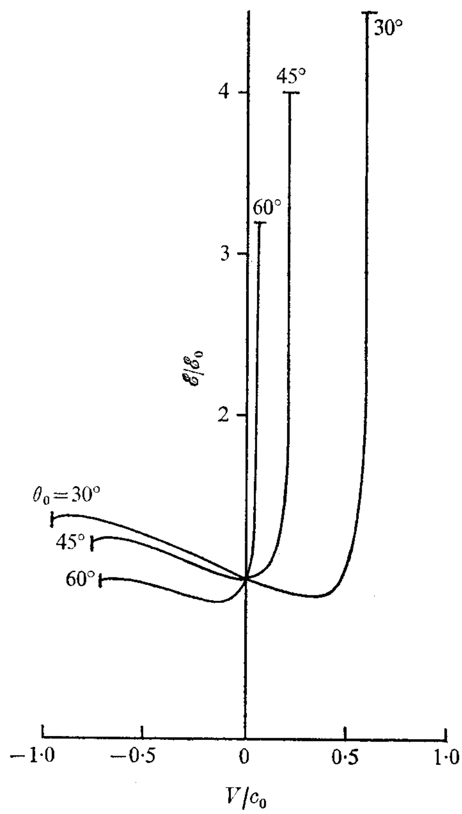

FIGURE 10. Energy density $\mathscr{E}$ as a function of stream velocity for the same situations as in figure 7.

5. The energy equation

In this section we find the form of the energy equation in terms of the averaged Lagrangian function

$$\mathscr{L}(\omega - U_i k_i, k_i), \tag{28}$$

where $i = 1, 2$, $U_1 = U$, $U_2 = V$, $k_1 = l$ and $k_2 = m$, in the notation of the previous sections. For unsteady flows with no main-stream flow Whitham (1965*b*, equation (39)) shows the energy equation to be

$$\frac{\partial}{\partial t}\left(\omega\frac{\partial \mathscr{L}}{\partial \omega} - \mathscr{L}\right) - \frac{\partial}{\partial x_i}\left(\omega\frac{\partial \mathscr{L}}{\partial k_i}\right) = 0, \tag{29}$$

in the case where his β and γ terms are absent, and where a repeated suffix implies summation. When there is a variable main stream U_i we have to include also convection terms $\partial(U_i\mathscr{E})/\partial x_i$ and 'radiation-stress' terms $S_{ij}\partial U_j/\partial x_i$, which express the interaction between the stream and the waves. Phillips (1966, equations (3.6.11) and (3.6.19), with the appropriate modifications for infinite depth) makes it clear that the components of S_{ij} are averaged momentum fluxes, which from Whitham's momentum equation (1965*b*, equation (40)) take the form

$$-(k_j\partial\mathscr{L}/\partial k_i - \mathscr{L}\delta_{ij}).$$

Thus the full energy equation is

$$\frac{\partial}{\partial t}\left((\omega - U_i k_i)\frac{\partial \mathscr{L}}{\partial \omega} - \mathscr{L}\right) + \frac{\partial}{\partial x_j}\left\{U_j\left((\omega - U_i k_i)\frac{\partial \mathscr{L}}{\partial \omega} - \mathscr{L}\right) - (\omega - U_i k_i)\frac{\partial \mathscr{L}}{\partial k_j}\right\} - \left(k_j\frac{\partial \mathscr{L}}{\partial k_i} - \mathscr{L}\delta_{ij}\right)\frac{\partial U_j}{\partial x_i} = 0, \tag{30}$$

where $\mathscr{L}$ is as in (28) but differentiations with respect to k_j are carried out *before* ω is replaced by $\omega - U_i k_i$.

It is now easy to establish the relation between this equation and Whitham's original equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial k_i}\right), \tag{31}$$

which we have already used in the steady case. Here $\mathscr{L}$ is as in (30) but the differentiation with respect to k_i is carried out *after* replacing ω by $\omega - U_j k_j$. It is only necessary to make the appropriate changes in $\partial\mathscr{L}/\partial k_i$ in (31) and to use the conservation of waves equations

$$\frac{\partial \omega}{\partial x_i} + \frac{\partial k_i}{\partial t} = 0, \quad \frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i} \tag{32}$$

and, for example,

$$\frac{\partial \mathscr{L}}{\partial x_i} = \frac{\partial \mathscr{L}}{\partial \omega}\frac{\partial}{\partial x_i}(\omega - U_j k_j) + \frac{\partial \mathscr{L}}{\partial k_j}\frac{\partial k_j}{\partial x_i}, \tag{33}$$

assuming $\mathscr{L}$ does not depend explicitly on x_i or t. If $\mathscr{L}$ depends on some other function of x_i, say a mean depth $h_0(x_i)$, this will be taken care of by further equations of the form

$$\partial\mathscr{L}/\partial h_0 = 0. \tag{34}$$

The result of all the algebra is that the energy equation (30) is equation (31) multiplied throughout by $kc = \omega - U_j k_j$. It is hoped to use (30) in a subsequent paper, but to include dissipation terms on the right-hand side and also to make some allowance for wave breaking. It should then be possible to consider the interactions between short gravity waves and longer waves considered as the basic flow. This will make an interesting comparison with the work of Longuet-Higgins (1969) and the recent paper by Hasselmann (1971).

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